

REPRESENTATION STABILITY FOR CONFIGURATION SPACES OF GRAPHS

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ABSTRACT. We consider for two based graphs G and H the sequence of graphs G_k given by the wedge sum of G and k copies of H . These graphs have an action of the symmetric group Σ_k by permuting the H -summands. We show that the sequence of representations of the symmetric group $H_q(\text{Conf}_n(G_\bullet); \mathbb{Q})$, the homology of the ordered configuration space of these spaces, is representation stable in the sense of Church and Farb. In the case where G and H are trees, we provide a similar result for glueing along arbitrary subtrees instead of the base point. Furthermore, we give similar stabilization results for configurations in spaces without any obvious action of the symmetric group.

1. INTRODUCTION

For a topological space X and a finite set S we define the *ordered configuration space of X with particles S* as

$$\text{Conf}_S(X) := \{f: S \rightarrow X \text{ injective}\} \subset \text{map}(S, X).$$

For $n \in \mathbb{N}$ we write $\mathbf{n} := \{1, 2, \dots, n\}$ and $\text{Conf}_n(X) := \text{Conf}_{\mathbf{n}}(X)$.

Let G be a finite connected graph, then we are interested in the homology of $\text{Conf}_n(G)$, the *ordered configuration space of n particles in G* . In [Lü14] we showed that at least one of $H^k(\text{Conf}_\bullet(G); \mathbb{Q})$ cannot be representation stable. In this paper we show that by stabilizing the *graph* instead of the number of particles we get representation stability.

Let us first specify what we mean by stabilizing the graph. For three graphs G_0, G_1 and H such that H is a subgraph of G_0 as well as of G_1 , denote by $G_0 \sqcup_H G_1$ the graph given by taking the disjoint union of G_0 and G_1 and glueing them together along H . Note that H is still a subgraph of $G_0 \sqcup_H G_1$, so we can iterate this construction. We write $G_0 \sqcup_H G_1^{\sqcup k}$ for the k -fold iterated glueing. The symmetric group Σ_k acts on this space by permuting the copies of G_1 .

Let G_0 be a finite graph and $H_i \subset G_i$ be pairs of finite graphs for $1 \leq i \leq \ell$ such that each H_i is also a subgraph of G_0 . Denote by $\underline{H} := \{H_1, \dots, H_\ell\}$ and $\underline{G} := \{G_0, \dots, G_\ell\}$. Let $\mathbf{G}[\underline{H}, \underline{G}]: \text{FI}^{\times \ell} \rightarrow \text{Top}$ be given by

$$\mathbf{G}[\underline{H}, \underline{G}](\mathbf{j}_1, \dots, \mathbf{j}_\ell) := G_0 \sqcup_{H_1} G_1^{\sqcup j_1} \dots \sqcup_{H_\ell} G_\ell^{\sqcup j_\ell}.$$

See Section 2 for more details. In this paper we prove that if we put certain restrictions on the graphs G_i and H_i , then the $\text{FI}^{\times \ell}$ -module

$$\mathbf{H}_{q,n}^{\mathbb{Z}}[\underline{H}, \underline{G}] := H_q(\text{Conf}_n(\mathbf{G}[\underline{H}, \underline{G}]); \mathbb{Z})$$

is finitely generated.

Theorem A. *If each graph H_i is a single point then $\mathbf{H}_{q,n}^{\mathbb{Z}}[\underline{H}, \underline{G}]$ is finitely generated in degree $(3n, 3n, \dots, 3n)$ for each $q, n \in \mathbb{N}$.*

Theorem B. *If all graphs G_i and H_i are trees, then $\mathbf{H}_{q,n}^{\mathbb{Z}}[\underline{H}, \underline{G}]$ is finitely generated in degree $(2n, 2n, \dots, 2n)$ for each $q, n \in \mathbb{N}$.*

Corollary 1.1. *In the same situation as in the theorems above, choose any non-decreasing (component-wise) functor*

$$F: \mathbf{FI} \rightarrow \mathbf{FI}^{\times \ell},$$

then the FI-module $\mathbf{H}_{q,n}^{\mathbb{Z}}[\underline{H}, \underline{G}] \circ F$ is finitely generated. In particular, the sequence

$$\mathbf{H}_{q,n}^{\mathbb{Q}} := H_q(\mathrm{Conf}_n(\mathbf{G}[\underline{H}, \underline{G}] \circ F); \mathbb{Q})$$

is representation stable and therefore the dimension of the sequence of vector spaces is eventually polynomial.

Corollary 1.2. *Let G, H be finite graphs with base point and define*

$$G_k := G \vee \underbrace{H \vee \dots \vee H}_{k \text{ times}}.$$

Then the FI-module $H_q(\mathrm{Conf}_n(G_{\bullet}))$ is finitely generated in degree $3n$. In particular, the sequence $H_q(\mathrm{Conf}_n(G_{\bullet}); \mathbb{Q})$ is representation stable and therefore the dimension of the sequence of vector spaces is eventually polynomial in k .

Remark 1.3. The fact that the dimension of this sequence is *bounded from above* by a polynomial can be seen more easily if G and H are trees: from [Ghr01, Theorem 2.6, p. 8] we know that the rank of the first homology of star graphs is polynomial in the number of edges. This implies that the size of the generating set for $H_q(\mathrm{Conf}_n(G_k))$ described in [CL16, Theorem 2, p. 2] is polynomial in k , giving a polynomial upper bound for the rank.

In the last section, we provide stabilization results for stabilization along an interval and the circle S^1 . For this, we take a based graph G whose base point has valence at least two. For each $k \in \mathbb{N}$, define the space $V_{\mathbf{k}}^I$ to be the interval with k copies of G wedged at $\frac{i}{k+1} \in [0, 1]$. Let $V_{\mathbf{k}}^{S^1}$ be given by the circle S^1 with k copies of G wedged at $\frac{i}{k+1} 2\pi \in S^1$. Then V_{\bullet}^I and $V_{\bullet}^{S^1}$ can be viewed as spaces over suitable categories, where morphisms induce maps mapping copies of G via the identity to other copies of G . See Figure 4 and Figure 7 for illustrations of such continuous maps.

Theorem C. *For each $n, q \in \mathbb{N}$, the $\mathbb{Z}[\tilde{\Delta}]$ -module*

$$W_{q,n,\bullet}^I := H_q(\mathrm{Conf}_n(V_{\bullet}^I); \mathbb{Z})$$

is finitely generated in degree n .

Theorem D. *For each $n, q \in \mathbb{N}$, the $\mathbb{Z}[\tilde{\Lambda}]$ -module*

$$W_{q,n,\bullet}^{S^1} := H_q(\mathrm{Conf}_n(V_{\bullet}^{S^1}))$$

is finitely generated in degree $6n$.

For the precise definition of the categories $\tilde{\Delta}$ and $\tilde{\Lambda}$ as well as the proof see Section 5.

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2. REPRESENTATION STABILITY AND $\mathrm{FI}^{\times \ell}$ -MODULES

In [CF13], Church and Farb introduced the concept of *representation stability*. We now recall the concept in the case of the symmetric group Σ_k , for more details see [CF13, Section 2.3, p. 19].

Let $\{V_k\}_{k \in \mathbb{N}}$ be a sequence of Σ_k -representations over \mathbb{Q} with linear maps

$$\phi_k: V_k \rightarrow V_{k+1}$$

which are homomorphisms of $\mathbb{Q}\Sigma_k$ -modules. Here we consider V_{k+1} as $\mathbb{Q}\Sigma_k$ module by the standard inclusion $\Sigma_k \hookrightarrow \Sigma_{k+1}$.

To describe stability for such a sequence, we need to compare Σ_k -representations to $\Sigma_{k'}$ -representations for $k' > k$. Recall that the irreducible representations of Σ_k over the rational numbers are in one to one correspondence to partitions λ of k . Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell \geq 0)$ of k , define for $k' - k \geq \lambda_1$ the irreducible $\Sigma_{k'}$ -representation $V(\lambda)_{k'}$ to be the one corresponding to the partition $(k' - k, \lambda_1, \dots, \lambda_\ell)$. Each irreducible representation can be written like this for a unique partition λ . For more details, see [CF13, Section 2.1, p. 14] and [FH91].

Definition 2.1 ([CF13, Definition 2.3, p.20]). The sequence $\{V_k\}$ is (*uniformly representation stable*) if, for sufficiently large k , each of the following conditions holds.

- $\phi_k: V_k \rightarrow V_{k+1}$ is injective.
- The $\mathbb{Q}\Sigma_{k+1}$ submodule generated by $\phi_k(V_k)$ is equal to V_{k+1} .
- Decompose each V_k into irreducible representations

$$V_k = \bigoplus_{\lambda} c_{\lambda,k} V(\lambda)_k$$

with multiplicities $0 \leq c_{\lambda,k} \leq \infty$. Then there exists an $N \geq 0$ such that for each λ , the multiplicity $c_{\lambda,k}$ is independent of $k \geq N$.

This reduces the description of the infinite sequence of Σ_k -representations to a finite calculation.

In [CEF15], Church-Ellenberg-Farb introduced the notion of FI-modules, which we now recall. Let FI be the category with objects all finite sets and morphisms all injective maps. We often consider the skeleton of this category given by the restriction to the finite sets $\mathbf{n} := \{1, \dots, n\}$ for $n \geq 0$.

Definition 2.2. Let R be a commutative ring. An $R[\mathrm{FI}]$ -module V_{\bullet} is a functor

$$V_{\bullet}: \mathrm{FI} \rightarrow R\mathrm{Mod}.$$

It is said to be *finitely generated in degree ℓ* if there exists a finite set X of elements in

$$\bigsqcup_{\substack{S \in \mathrm{FI} \\ |S| \leq \ell}} V_S,$$

such that the smallest sub-FI-module containing all these elements is V_{\bullet} . Here, $|S|$ is the cardinality of S .

For sequences of finite dimensional representations, the notion of a finitely generated FI-module is a generalization of representation stable sequences by the following result:

Theorem 2.3 ([CEF15, Theorem 1.13, p. 8]). *An FI-module V_\bullet over a field of characteristic 0 is finitely generated if and only if the sequence $k \mapsto V_k$ is representation stable and each V_k is finite dimensional.*

This result reduces the uniform decomposition of the representations V_k to finding a finite set of generators. Furthermore, Church-Ellenberg-Farb proved that the dimension of representation stable sequences grows polynomially:

Theorem 2.4 ([CEF15, Theorem 1.5, p. 4]). *Let V_\bullet be an FI-module over a field of characteristic 0. If V_\bullet is finitely generated then the sequence of characters χ_{V_\bullet} is eventually polynomial. In particular, $\dim V_k$ is eventually polynomial in k .*

In order to describe stabilization in multiple directions we look at the product category $\mathrm{FI}^{\times \ell}$ consisting of $\ell \geq 1$ copies of the category FI. An $\mathrm{FI}^{\times \ell}$ -module is then a functor $\mathrm{FI}^{\times \ell} \rightarrow \mathrm{RMod}$, the notion of finite generation is defined analogously.

To define such a module, it is sufficient to define it on the skeleton consisting of the objects $(\mathbf{j}_1, \dots, \mathbf{j}_\ell)$ for $j_i \in \mathbb{N}$ and the morphisms between them. In the introduction we defined $\mathbf{G}[\underline{H}, \underline{G}]$ for those objects. To define the images of morphisms, notice that each summand of G_i can be labeled by a number between 1 and j_i . For a map $\phi: \mathbf{j}_i \hookrightarrow \mathbf{j}'_i$ we define the induced map to send the summand with label $m \in \mathbf{j}_i$ to the summand with label $\phi(m)$ via the identity.

Clearly, if V is a finitely generated $\mathrm{FI}^{\times \ell}$ -module and $F: \mathrm{FI} \rightarrow \mathrm{FI}^{\times \ell}$ is any non-decreasing functor, then the FI-module $F^*V := V \circ F$ is finitely generated: since F is non-decreasing, each component of F is either eventually constant or unbounded.

3. GENERATORS FOR THE HOMOLOGY OF CONFIGURATION SPACES OF GRAPHS

In this section, we give an overview over the results of [CL16] that we need in this paper. This includes a generating set for the homology of configuration spaces of trees and a description of how to turn those into generating sets for the non-tree case.

3.1. Configurations in trees. We need the following definition from the mentioned paper.

Definition 3.1. A representative S of a homology class $\sigma \in H_q(\mathrm{Conf}_n(G))$ is called a *product of the cycles S_1 and S_2* for $[S_1] \in \mathrm{Conf}_{T_1}(G)$ and $[S_2] \in \mathrm{Conf}_{T_2}(G)$ with $T_1 \sqcup T_2 = \mathbf{n}$ if the image of (S_1, S_2) under the product map

$$\mathrm{Conf}_{T_1}(G) \times \mathrm{Conf}_{T_2}(G) \rightarrow G^n$$

is equal to the image of S under the inclusion $\mathrm{Conf}_n(G) \subset G^n$.

For $k \geq 3$ let Star_k be the star graph with k leaves and H be the tree with two vertices of valence three. We call an element $\sigma \in H_1(\mathrm{Conf}_n(G))$ *basic* if there exists a piecewise linear embedding ι of Star_k for some k or H into G such that σ is in the image of the induced map $H_1(\iota)$.

We will use the following result:

Theorem 3.2 ([CL16, Theorem 2, p. 2]). *Let G be a finite tree and n a natural number. Then the homology of $\text{Conf}_n(G)$ in degree $q \geq 0$ is generated by products of basic cycles.*

Remark 3.3. We will also use that the embeddings of H can be chosen such that they contain precisely two essential vertices, which can be arranged by splitting an H -graph containing k vertices into $k - 1$ of them, each containing exactly two vertices. Also, note that after fixing those two vertices, we can choose the edges of the embedded H -graph arbitrarily: in the proof of the theorem above we only needed that the valence of the vertices is at least three. The cycles given by different choices of edges differ by cycles in the stars of the corresponding vertices.

3.2. Glueing two leaves of a graph. In order to prove Theorem A, we need to understand how to turn a set of generators for $H_q(\text{Conf}_n(G))$ into a set of generators for $H_q(\text{Conf}_n(\overline{G}))$, where \overline{G} is G with two of its leaves glued together.

Definition 3.4. Let G be a graph and W a subset of the vertices. The *configuration space of G with sinks W* is defined as

$$\text{Conf}_n^{\text{sink}}(G, W) = \{(x_1, \dots, x_n) \mid \text{for } i \neq j \text{ either } x_i \neq x_j \text{ or } x_i = x_j \in W\} \subset G^n.$$

This definition is useful since it allows comparing configurations in G to configurations in a quotient of G by turning the collapsed part into a sink. This technique can be used to investigate the local geometry of the configuration space by collapsing large parts of the graph. For a typical example of a loop in $\text{Conf}_n^{\text{sink}}([0, 1], \{0, 1\})$, see Figure 1.

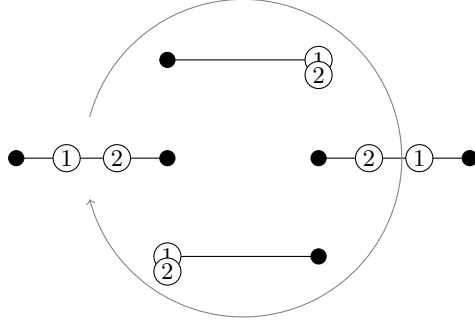


FIGURE 1. A 1-cycle in the configuration space of the interval with two sinks.

In this paper, we will use the same combinatorial model of the configuration space of a graph (with or without sinks) as in [CL16]. We only briefly sketch the construction, for more details see [CL16] and [Lü14].

The combinatorial model is a cube complex. We have a 0-cube for each configuration where for each edge e which is occupied by $\ell \geq 1$ particles the positions of the particles are $\frac{i}{\ell+1} \in [0, 1] \cong e$ for $1 \leq i \leq \ell$. A k -cube for $k > 0$ is given by choosing such a 0-cell and k particles of the corresponding configuration, each sitting at the first or last position of some edge. Moving along one of the axes of the cube then corresponds to moving the chosen particle linearly to the vertex. We only allow those k -cubes such that no two particles approach the same non-sink vertex.

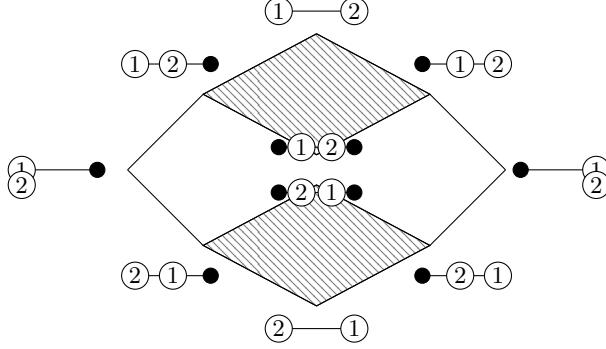


FIGURE 2. The combinatorial model of $\text{Conf}_2^{\text{sink}}(I, \{0, 1\})$. It is homotopy equivalent to a 1-dimensional complex by pushing the two 2-cells towards the outer embedded S^1 .

For an example, see Figure 2.

In the proof of the general case of Theorem 1 of [CL16], it was shown that the homology of $\text{Conf}_n(\overline{G})$ is generated by pairs $([S], \sigma)$ with $[S]$ a generator of the homology of $\text{Conf}_P(G)$ for $P \subset \mathbf{n}$ and σ a homology class of the configuration space of the particles $\mathbf{n} - P$ in the interval with zero, one or two sinks or the circle S^1 with one sink, depending on the representatives of the class $[S]$. In the construction of these pairs, we started with the cycle S and constructed pairs $([S], \sigma)$ via a spectral sequence argument. If $[S]$ has a representative avoiding a terminal vertex of the glued edge, we turn the corresponding vertex of the unit interval into a sink. This sink represents the ability to reorder the particles of σ on the star of the corresponding vertex of \overline{G} *disjointly from the support of a representative of $[S]$* .

We now make a case distinction on the type of the generator σ to describe how to construct the homology classes in $H_*(\text{Conf}_n(\overline{G}))$ corresponding to such pairs.

σ a standard generator in degree zero: If $\sigma \neq 0$ is a zero cycle, then this pair just corresponds to the image of $[S]$ under

$$H_*(\text{Conf}_P(G)) \rightarrow H_*(\text{Conf}_P(\overline{G}))$$

induced by the canonical *inclusion* $G \hookrightarrow \overline{G}$, where we add the remaining particles to the glued edge in the order given by any representative of σ .

σ a standard generator in degree 1 in the interval with two sinks: Every such class can be represented by a cycle η where all particles sit on the edge and reorder themselves in an alternating way on the left and right sinks, c.f. Figure 1. We think about such 1-cycles as having vertices where all particles sit on the interior of the interval and edges moving all of them to one of the sinks and back. The cycle corresponding to the pair $([S], \sigma)$ is constructed as follows: start on some vertex of η , then the corresponding configuration has all particles on the interior of the interval. For each 1-cell T of η describing a reordering on the sink corresponding to some vertex $v \in \overline{G}$, choose a cycle \tilde{S} avoiding v with $[S] = [\tilde{S}]$ and a chain X with boundary $S - \tilde{S}$. The reordering T has two configurations T_0 and T_1 of particles on the interval, one before and one after the reordering. Denote by X_{T_i} the chain X where in each cell we add the particles $\mathbf{n} - P$ to the glued edge in the order given

by T_i . Now glue together X_{T_0} , X_{T_1} and the product of \tilde{S} and a reordering path γ of the particles $\mathbf{n} - P$ in the star of v , c.f. Figure 3. Putting these together for all reorderings yields the homology class represented by this pair.

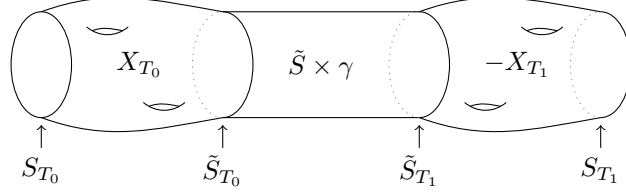


FIGURE 3. Constructing a piece of the cycle represented by a pair of cycles $[S]$ and σ .

σ a standard generator in degree 1 in the circle with one sink: These 1-cycles are generated by cycles where all particles sit on the edge, then all but one of the outmost particles p move to the sink, p goes around the circle once and all other particles return to their initial position. To get the generator corresponding to this pair, we do the construction analogous to the previous case: represent S by \tilde{S} freeing one of the vertices v , move the particles off the glued edge onto the star of v , choose a chain moving p to the other vertex of the glued edge and move all particles back to the glued edge. Then, piece together the chains in the obvious way.

To summarize, we can find a generating set for $H_*(\text{Conf}_n(\overline{G}))$ by taking a generating set for $H_*(\text{Conf}_k(G))$ for all $k \leq n$, adding the remaining $n - k$ particles to the glued edge in different ways and choosing chains between homologous cycles of this form. For more details, see [CL16].

4. THE PROOF OF THEOREM A AND THEOREM B

Definition 4.1. Let C be a subspace of $\text{Conf}_n(G)$ for a finite graph G and H a subspace of G . Then we say that C is *supported in H* if for each point $x \in C$ all particles x_i are on H . We say that a homology class $\sigma \in H_q(\text{Conf}_n(G))$ is *supported in H* if there exists a representative of σ which is supported in H .

We first prove Corollary 1.2 for star graphs by hand.

Proposition 4.2. *Corollary 1.2 is true for G the point and H the interval $[0, 1]$ with 0 as base point. In fact, the homology is generated by cycles meeting at most $n + 3$ many copies of H , so the FI-module is generated in degree $n + 3$.*

Remark 4.3. For $n = 2$ the argument presented below is easily modified to show that $H_1(\text{Conf}_2(G_\bullet))$ is generated in degree $n + 2 = 4$. Since $n + 3 \leq 2n$ for $n > 2$, this shows that $H_1(\text{Conf}_n(G_\bullet))$ is generated in degree $2n$, which will be used in the proof of Theorem B.

Proof of Proposition 4.2. The combinatorial model of this configuration space is a graph (c.f. [Ghr01, Theorem 2.6, p. 8], [Lü14]), so we only need to consider 1-cycles. Choose any subgraph $\text{Star}_3 \subset \text{Star}_k$. Assume that we have a connected 1-cycle σ which visits each vertex of the configuration space at most once. The claim is that

we can write σ as a sum of cycles where each particle uses at most one edge outside of Star_3 .

Let p be a particle and choose a vertex v of σ where p sits on the vertex of the star. If this does not exist, then p is fixed and therefore uses at most one edge. Now follow the cycle until p sits on the vertex again and the next edge would move p onto the *second* leaf of $\text{Star}_k - \text{Star}_3$, we call the corresponding vertex w . Now choose the following path γ back to v during which p always stays on Star_3 : move p onto an edge e_1 of Star_3 and keep it there. Follow σ back ignoring the movement of p and using the connectedness of the configuration space of Star_3 to move p out of the way if other particles need to move along e_1 .

This decomposes σ into two parts: the segment of σ between v and w followed by γ , in which p visits only one edge not in Star_3 , and γ followed by the remaining segment of σ . Continuing this process, we get a sum decomposition of σ where p visits at most one additional edge in each summand.

Since we did not increase the number of edges visited by any other particle, we can repeat this for every p and get a sum decomposition of σ of the required form.

Consequently, for each $N \geq n+3$ we can generate $H_1(\text{Conf}_n(G_N))$ by cycles such that each one of them is supported in some subgraph $\text{Star}_{n+3} \hookrightarrow G_N$. Therefore, the $\mathbb{Z}\Sigma_N$ -span of the image of the map

$$H_1(\text{Conf}_n(G_{n+3})) \rightarrow H_1(\text{Conf}_n(G_N))$$

is the whole module and the FI-module $H_1(\text{Conf}_n(G_\bullet); \mathbb{Z})$ is finitely generated in degree $n+3$. \square

Proof of Theorem B. Let $n > 1$ and (k_1, \dots, k_ℓ) be such that each k_i is at least $2n$ and assume that none of the G_i for $i > 0$ is equal to the point. By Theorem 3.2, the homology of $\text{Conf}_n(\mathbf{G}(k_1, \dots, k_\ell))$ is generated by products of basic cycles, since the graph is a tree. By Remark 3.3 we can assume that the embedded H -graphs contain exactly two vertices because $k_i > 3$ and therefore the valence of all vertices is at least three.

If an H -generator involves a vertex in one of the H_i , then we can arrange that it only involves edges of $\mathbf{G}(3, 3, \dots, 3) \subset \mathbf{G}(k_1, \dots, k_\ell)$ by Remark 3.3 again.

Each star generator with vertex on some H_i with k particles can be written as a sum of generators each using only $2k$ different edges by Proposition 4.2 and Remark 4.3. Therefore, each summand visits at most $2k$ distinct copies of each of the G_i . Every star generator disjoint from all H_i meets at most one of the copies of one of the G_i .

Hence, we can generate the whole homology by generators which each meet at most $2n$ different copies of each of the G_i . This implies that the $\mathbb{Z}[\Sigma_{k_1} \times \dots \times \Sigma_{k_\ell}]$ -span of the image of

$$H_*(\text{Conf}_n(\mathbf{G}(2n, \dots, 2n))) \rightarrow H_*(\text{Conf}_n(\mathbf{G}(k_1, \dots, k_\ell)))$$

is the whole module, finishing the proof. \square

Proof of Theorem A. The restriction that each H_i is the point means that the glueing along H_i corresponds to a wedge sum for the corresponding choices of base points. If all G_i are trees this is true by Theorem B. We will now prove that the theorem remains true if we glue leaves together.

Let (k_1, \dots, k_ℓ) be such that each k_i is at least $3n$ and assume that none of the G_i for $i > 0$ is equal to the point. Now assume that all G_i are trees, then we

know that we can generate the homology $\mathbf{H}_{q,n}^{\mathbb{Z}}[\underline{H}, \underline{G}](k_1, \dots, k_\ell)$ by cycles where the particles visit at most $2n$ copies of each of the G_i . If we now glue two leaves of one of the copies of G_{i_0} for some i_0 , then we know from Section 3.2 how to construct a generating set for the corresponding homology.

Let $([S], \sigma)$ be a pair of cycles, where S consists of m particles and meets at most $2m \leq n$ copies of each G_i . If σ is a 0-cycle then the cycle represented by this pair meets at most one copy of G_i more than S , so in particular it still meets in total at most $2n$ copies of each G_i (if it meets one more copy of G_i , then $m < n$ and therefore $2m + 1 \leq 2n - 1$). It remains to handle the case where σ is a 1-cycle.

Choose for each i an edge $e_i \subset G_i$ at the base point H_i . For each i and each particle p , let G_i^p be a copy of G_i such that S is supported outside of $G_i^p - H_i$ and $G_i^p \neq G_i^{p'}$ for $p \neq p'$. Denote by e_i^p the edge in G_i^p corresponding to $e_i \subset G_i$. In the construction of the generator corresponding to $([S], \sigma)$ we then choose for each 1-cell T of a representative of σ a $(q+1)$ -chain Y_T bounding S_{T_0} and S_{T_1} , see Figure 3. We now describe how to turn Y_T into a chain Y'_T which is supported in the union of G_0 , all G_i^p and the copies of the G_i in the support of S .

For each particle p define the continuous map

$$\phi_p^S: \mathbf{G}[\underline{H}, \underline{G}](k_1, \dots, k_\ell) \rightarrow \mathbf{G}[\underline{H}, \underline{G}](k_1, \dots, k_\ell)$$

which is the identity on G_0 and all copies of all G_i meeting the support of S . On a copy of G_i *outside* the support of S , define the map to be given by

$$x \mapsto \min\{d(x, H_i), 1/2\} \in [0, 1] \cong e_i^p \subset \mathbf{G}[\underline{H}, \underline{G}](k_1, \dots, k_\ell),$$

where the isomorphism is chosen such that 0 corresponds to the vertex H_i and the path metric on the graph is such that each edge has length 1. The image of this map is by definition contained in the union of G_0 , all G_i^p and all copies of the G_i meeting the support of S . Now define the continuous map

$$\phi^S: \text{Conf}_n(\mathbf{G}(k_1, \dots, k_\ell)) \rightarrow \text{Conf}_n(\mathbf{G}(k_1, \dots, k_\ell))$$

to be given by mapping each particle p via ϕ_p . Every particle p is either mapped via the identity or lands on $G_i^p - H_i$, and since $G_i^p - H_i$ is disjoint from $G_i^{p'} - H_i$ for $p \neq p'$, the map is well-defined.

By definition, $\phi^S(S_{T_j}) = S_{T_j}$ for $j \in \{0, 1\}$, so $Y'_T := \phi^S(Y_T)$ has the required properties. Glueing together these chains gives a cycle supported in the union of G_0 and at most $3n$ copies of each G_i .

Since each of the generators we construct in this way already comes with a choice of copies G_i^p for each p and i , we can repeat the same argument for any subsequent glueing.

By glueing together leaves of all G_i 's we see inductively that the module $\mathbf{H}_{q,n}^{\mathbb{Z}}[\underline{H}, \underline{G}](k_1, \dots, k_\ell)$ is generated by cycles meeting at most $3n$ copies of each of the G_i , which proves the theorem. \square

Remark 4.4. In the proof above, we always replace the movement of a particle p inside any new copy of G_i by *constant* movement in $e_i^p \subset G_i^p$ instead of the “same” movement in G_i^p since for $i = i_0$ we don't know whether the leaves of G_i^p are already glued or not, so the “same” movement may not be well-defined in G_i^p .

5. FINITE GENERATION OVER SIMILAR CATEGORIES

In this section we provide examples for spaces over categories different from $\mathbf{FI}^{\times \ell}$ whose homology groups stabilize in a similar sense.

5.1. Stabilization along the interval. Let $\tilde{\Delta}$ be the category with objects finite totally ordered sets and morphisms injective order preserving maps. Functors from this category can be given by defining them on the skeleton having $\mathbf{k} := \{1, \dots, k\}$ with the canonical ordering as objects.

Let (G, b_G) be a based finite graph with $\text{val}(b_G) \geq 2$ and I the unit interval. Then let $V_{\bullet}^I: \tilde{\Delta} \rightarrow \text{Top}$ be the functor sending \mathbf{k} to k copies of G wedged to the unit interval at the points $\frac{i}{k+1} \in I$ for $1 \leq i \leq k$. The image of a morphism $\phi: \mathbf{k} \hookrightarrow \mathbf{k}'$ is given by the canonical map stretching the parts of the interval between the copies of G and mapping the copies of G by the identity to the copies of G according to ϕ , c.f. Figure 4.

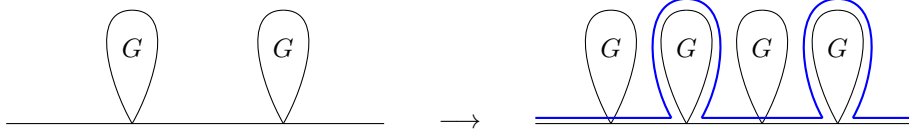


FIGURE 4. The map induced by $\phi: \mathbf{2} \rightarrow \mathbf{4}$ given by $\phi(1) = 2$ and $\phi(2) = 4$.

Now Theorem C says that the module

$$W_{q,n,\bullet}^I := H_q(\text{Conf}_n(V_{\bullet}^I); \mathbb{Z})$$

is finitely generated. For the proof we need the following result.

Lemma 5.1. *Let $H \subset G$ be two graphs such that each vertex of H incident to an edge of $G - H$ has valence at least three and no pair of vertices of H is connected via a path in $G - H$. Let S be a q -cycle in $\text{Conf}_n(G)$ representing the zero homology class. Then we can find a $(q+1)$ -chain X with boundary S such that every particle that stays on H for each cell of S also stays on H for each cell of X .*

Proof. First, we show this for the case where $G = H \vee G_0$ and denote the base point by b_G . Let X be some chain with boundary S . We now replace X by a chain with the properties described in the statement. Let P be the set of particles that stay on H for each cell of S .

Define the continuous map

$$\phi: \text{Conf}_n(H \vee G_0) \rightarrow \text{Conf}_n\left(H \vee G_0 \vee \bigvee_{p \in P} [0, 1]_p\right)$$

to be given by the inclusion for the particles $\mathbf{n} - P$ and

$$x \mapsto \begin{cases} x & \text{if } x \in H, \\ \min\{d(x, b_G), 1/2\} \in [0, 1]_p & \text{else.} \end{cases}$$

for each particle $p \in P$, similar to the map ϕ in the proof of Theorem A. This gives a chain $X' := \phi(X)$ bounding $\phi(S)$ such that each particle $p \in P$ stays on $H \vee [0, 1]_p$.

Choose three edges e_1, e_2, e_3 at b_G belonging to H . We now define a map

$$\psi: \text{Conf}_n \left(H \vee G_0 \vee \bigvee_{p \in P} [0, 1]_p \right) \rightarrow \text{Conf}_n(H \vee G_0)$$

by giving the images of cells C of the combinatorial model of the domain of ψ via the following case distinction:

All moving particles of C are on G : First, assume that no moving particle of C moves from e_1 to b_G . Map C to the cell C' where we put the particles sitting on any $[0, 1]_p$ onto the edge e_1 ordered alphabetically first by the natural order on $P \subset \mathbf{n}$ and second by the order of the particles on the corresponding intervals. Put these particles onto e_1 in such a way that there are no other particles between them and the base point.

If in C a particle p_0 moves from e_1 to b_G , take the face D of C where p_0 is fixed on the interior of e_1 and map it to the cell D' constructed like in the previous paragraph. To define the map on all of C , map the movement of p_0 to the following: move the set of particles W sitting between p_0 and b_G to e_2 , then p_0 to e_3 , followed by moving the particles in W back to e_1 . Finally, move p_0 to the base point b_G , c.f. Figure 5. This is well-defined since no other particle moves towards b_G by the definition of the combinatorial model.

The particle p_0 moves off the edge $[0, 1]_p$: take the face of C where p_0 is fixed on the interior of $[0, 1]_p$ and map this face according to the previous case. To describe the map on the whole cube C , map the movement of p_0 to the movement given by precisely the same description as in the previous paragraph.

It is straightforward to check that this definition of ψ on individual cubes is compatible with taking faces, so this defines a continuous map on the whole configuration space.

Since $\psi(\phi(S)) = S$, this defines a chain $X'' := \psi(\phi(X))$ such that each particle of P always stays in H .

For the general case, observe that the assumption that no pair of vertices of H is connected via a path in $G - H$ implies that G can be constructed from H by performing wedge sums. Now construct maps ϕ and ψ analogously for this simultaneous wedging of multiple graphs. \square

Corollary 5.2. *Let G, H be finite based graphs and $n \in \mathbb{N}$. If the base point b_G of G has valence at least three, then the map*

$$H_q(\text{Conf}_n(G)) \rightarrow H_q(\text{Conf}_n(G \vee H))$$

induced by the inclusion $\iota: G \hookrightarrow G \vee H$ is injective for all $q \geq 0$.

Proof of Theorem C. Consider $V_{\mathbf{k}}^I$ for $k \geq 3n$. If G is a tree, then by Theorem 3.2 the homology is generated by products of basic cycles. These can be chosen such that each such generator visits at most n copies of $G - \{b_G\}$. Therefore, the result is true in this case.

We now perform induction over the rank of the whole graph $V_{\mathbf{k}}^I$ (handling all n at the same time), so we need to show that after glueing two leaves of one of the graphs wedged to the interval the homology is still generated by cycles supported in $2n$ wedge summands.

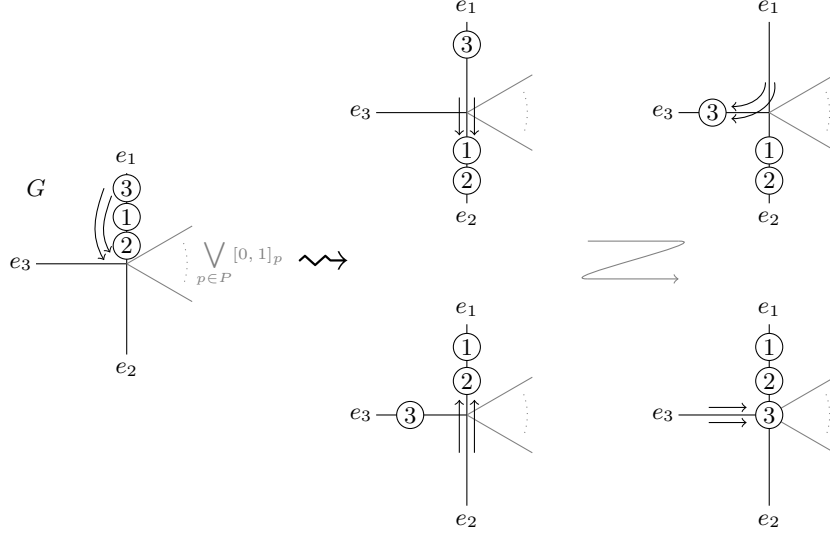


FIGURE 5. Mapping a cell where particle $p = 3$ moves to v along e_1 and the particles 1 and 2 are parked on e_1 .

Assume we are given the graph $V_{\mathbf{k}}^I$ with possibly some of the leaves already glued together. Denote by G_i for $1 \leq i \leq k$ the i -th wedge summand of this graph. Notice that G_i is not necessarily the same as G_j for $i \neq j$. Now let i_0 be the index of the graph G_{i_0} whose leaves we glue next. We use the description of generators for the homology of configurations in the glued graph of Section 3.2 again.

Let $([S], \sigma)$ be a pair of cycles and let k be the number of particles of S . Then, we can assume by induction that S meets at most $2k$ of the G_i . If $k = n$, then the pair just corresponds to the cycle S considered as a cycle in the glued graph and therefore meets at most $2n$ of the G_i , so we can assume $k < n$. If σ is a 0-cycle, then the corresponding cycle in the glued graph meets at most $2k + 1 \leq 2(n - 1) + 1 < 2n$ of the G_i .

It remains to handle the case that σ is a 1-cycle. To construct the cycle corresponding to this pair we choose a representative of σ and realize each 1-cell T of this representative with a chain Y_T bounding S_{T_0} and S_{T_1} , see Figure 3.

If S is supported outside of G_{i_0} , then Y_T can obviously be chosen to use at most one G_i more than S , namely G_{i_0} . Otherwise, take the nearest two graphs G_ℓ, G_r of G to the left and right of G_{i_0} which intersect the support of S only in their base point or not at all. Take the smallest connected subgraph H_{S, i_0} containing all G_i for $\ell \leq i \leq r$, see Figure 6.

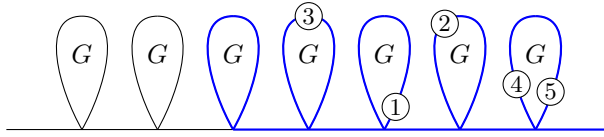


FIGURE 6. The subgraph $H_{S,6}$.

We can now write $V_{\mathbf{k}}^I$ as $H_\ell \vee_{b_{G_\ell}} H_{S,i_0} \vee_{b_{G_r}} H_r$ for some graphs H_ℓ, H_r . If H_ℓ contains at least one copy of G then the valence of H_{S,i_0} at b_{G_ℓ} is at least three, and the same holds for H_r and b_{G_r} . Lemma 5.1 shows that we can now replace Y_T by a chain such that all particles that start in H_{S,i_0} also stay in H_{S,i_0} . Keeping the particles in H_ℓ and H_r fixed, this yields a chain Y'_T which meets at most two of the G_i more than S , namely G_l and G_r . Therefore, the pair corresponds to a cycle meeting at most $2k + 2 \leq 2(n - 1) + 2 = 2n$ of the G_i . \square

5.2. Stabilization along the circle. Let $\tilde{\Lambda}$ be the category with objects finite cyclically ordered sets and morphisms injective maps preserving the cyclic order. Now we do the same construction as in the previous section, only this time we attach along the circle S^1 .

Define the functor $V_{\bullet}^{S^1} : \tilde{\Lambda} \rightarrow \text{Top}$ to be given on the sets \mathbf{k} (with the standard cyclic ordering) by the circle S^1 with k copies of G wedged to it at the angles $\frac{i}{k+1}2\pi$ for $1 \leq i \leq k$. Here we chose an arbitrary order representing the cyclic ordering, two different choices differ only by a rotation. Morphisms get mapped to continuous maps stretching the interval pieces of the S^1 and map the copies of G by the identity to each other, c.f. Figure 7.

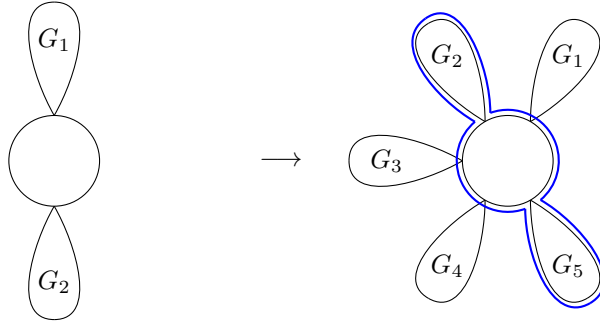


FIGURE 7. The map induced by $\phi: \mathbf{2} \rightarrow \mathbf{5}$ given by $\phi(1) = 2$ and $\phi(2) = 5$. The graph G_i is G with label i .

Proof of Theorem D. We prove this theorem by cutting open each $V_{\mathbf{k}}^{S^1}$ yielding $V_{\mathbf{k}}^I$. To construct a generating set for $W_{q,n,\mathbf{k}}^{S^1}$ we use Section 3.2 again for the glueing of 0 and 1 of the unit interval inside $V_{\mathbf{k}}^I$. Thus, we need to consider pairs of cycles $([S], \sigma)$, where S has ℓ particles. By the previous theorem, we can assume that S meets at most 2ℓ copies of G .

Now given a chain Y_T as in the proof of Theorem C, choose for each copy of $G - \{b_G\}$ intersecting with the support of S the nearest copies of $G - \{b_G\}$ to the left and right which are disjoint from the support of S . Then, use Lemma 5.1 again to show that we can arrange that the support of Y_T is given by the copies of G meeting the support of S and these chosen copies of $G - \{b_G\}$.

If $\ell = n - 1$, then T can be represented in a way that the single particle p only moves along the interval, so the corresponding cycle meets at most $(1 + 2)2(n - 1) = 6n - 6 \leq 6n$ copies of G .

For $\ell < n - 1$, we can represent T such that its particles only meet the first and last copy of G , so that the cycle represented by the pair meets at most $(1 + 2)2(n - 2) + 2 = 6n - 4 \leq 6n$ copies of G . \square

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